Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 1, pp. 1–12, 2002

THE SIMPLEST GALILEAN-INVARIANT AND THERMODYNAMICALLY CONSISTENT CONSERVATION LAWS

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UDC 517.9:539.3

This paper gives an introduction to formalization of Galilean-invariant and thermodynamically consistent equations of mathematical physics in which unknowns are transformed in rotations by irreducible representations of integer weights. This formalization is based on the theory of representations of the group $SO(3)$.

Introduction. Investigation of thermodynamically consistent equations and systems used in problems of continuum mechanics and physics was begun in the sixties of the 20th century [1, 2]. Originally they were used to construct examples of solutions. More recently, the number of problems studied with the help of such equations has increased and the systems of equations have become more and more complex (see [3–14]). In this connection, attempts have been made to use the techniques of group representation theory to describe rotationally-invariant thermodynamically consistent systems [15, 16]. However, these attempts have not yet resulted in a transparent theory.

In the present paper, which is a continuation of the group analysis of partial differential equations [12–16], we study only the simplest thermodynamically consistent equations, with emphasis on detailed investigation of their invariance under Galilean transformations of coordinate systems. Such transformations are a superposition of a conversion to a coordinate system that moves at constant velocity and an orthogonal transformation of spatial coordinates. In this case, unknown vector-functions are transformed with the help of orthogonal representations of the rotation group $SO(3)$ and spatial reflections. In the present paper, we consider only rotations, and therefore, we do not distinguish between vectors and pseudovectors, which respond differently to rotations.

The tensor variables used in mechanics are generally transformed in rotations by representations of a rather complex structure, and they can be decomposed into the simplest irreducible representations. It should be noted that for tensors of the third and higher orders, such a decomposition is not unique. Therefore, much attention is given to the symbolism connected with the use of irreducible representations.

As an example, we give the decomposition of an arbitrary orthogonal tensor of the second order into irreducible components. Such a tensor consists of nine elements, which fill a 3×3 matrix, and splits into three tensor terms: a diagonal matrix with equal diagonal elements $a = (a_{11} + a_{22} + a_{33})/3$, a skew-symmetric tensor, and a symmetric tensor with zero trace.

The quantity a is a scalar: it is not changed by rotations of the coordinate system. The three nonzero elements of the skew-symmetric tensor is transformed as a 3D vector. The matrix of elements from the 5D linear space of symmetric tensors of the second order with zero trace is conveniently written as

$$a_{-2}\begin{pmatrix} 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 \end{pmatrix} + a_{-1}\begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ a_0\begin{pmatrix} -1/\sqrt{6} & 0 & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & -1/\sqrt{6} \end{pmatrix} + a_1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix} + a_2\begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} \end{pmatrix}.$$

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Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 1, pp. 3–16, January–February, 2002. Original article submitted June 28, 2001.

In rotations, the vector $(a_{-2}, a_{-1}, a_0, a_1, a_2)^{\mathrm{t}}$, composed of the coefficients a_j, is transformed by the 5D irreducible orthogonal representation of the rotation group.

All the conservation laws studied in this paper reduce to symmetric Friedrichs-hyperbolic equations. In such conservation laws, dissipative terms, i.e., viscous friction and diffusion, are ignored. To take these into consideration, the conservation laws should be modified. Here we give just one example of such modification for a scheme of modeling thermal relaxation in a multitemperature medium.

All papers on the theory of representations of the rotation group [17–22] of which we are aware consider only complex matrix elements of unitary representations of this group, whereas applications to problems of classical mechanics should be based on real matrices of orthogonal representations, whose matrix elements are given in Sec. 3. To find these elements, elementary but rather cumbersome calculations were made, which, in essence, repeat the scheme used in the theory of unitary representations (see [23]).

We believe that the present research is of interest for both mathematicians and specialists in applied fields, and the scheme proposed here can be extended to more complex equations and the equations of relativistic theory, in which thermodynamically consistent conservation laws are also widely used.

1. Description of the "Simplest" System and Its Preliminary Analysis. The aim of this paper is to describe a formal general scheme that covers many well-known Galilean-invariant systems of differential equations of phenomenological mathematical physics, which contain both the laws of conservation of mass, momentum, and energy and the law of increase (or conservation) of entropy. In writing each such system, we use the governing "thermodynamic potential" L that results from systematization of the various thermodynamic potentials appearing in concrete physical problems. The required functions are the variables q_0, u_1, u_2, u_3, q_1, $q_2, \dots$, and T on which this potential depends:

$$L = L(q_0, u_1, u_2, u_3, q_1, q_2, \dots , T).$$

It should be noted that in this paper, such "thermodynamic" variables also include the components u_1, u_2, and u_3 of the velocity $\boldsymbol{u} = \boldsymbol{u}(x_1, x_2, x_3, t)$ at which points of a continuum move. The variable T is the temperature of the medium and the density is characterized by the quantity $L_{q_0} = L_{q_0}(q_0, u_1, u_2, u_3, q_1, q_2, \dots , T)$ which is related to the variable q_0 by means of the potential L.

Having specified the potential L, we can write a simple standard system of equations, whose properties are then studied in detail. We indicate which equations of the system describe the conservation laws, and under what constraint (on the thermodynamic potential) the systems are Galilean-invariant and ensure the (local) well-posedness of the Cauchy problem. At the end of this section, we give possible generalizations in which forces of viscosity, diffusion, etc., described by second-order derivatives are included in the equations. We now proceed to preliminary analysis of this "simplest" system. In conversion to a coordinate system moving at constant velocity relative to the initial coordinate system, the invariance of the equations chosen as the "simplest" is among the questions discussed in this section, and the invariance under rotations is considered in Sec. 2.

As the "simplest" systems, we choose systems of equations in the form

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L)_{q_0}}{\partial x_k} = 0, \tag{1.1a}$$

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} = 0, \tag{1.1b}$$

$$\frac{\partial L_{q_\gamma}}{\partial t} + \frac{\partial (u_k L)_{q_\gamma}}{\partial x_k} = -\varphi_\gamma, \tag{1.1c}$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial (u_k L)_T}{\partial x_k} = \frac{q_\gamma \varphi_\gamma}{T} \tag{1.1d}$$

(summation over repeated indices k and γ is performed). The divergent equations (1.1a) and (1.1b) model the laws of conservation of mass and momentum. The variables q_γ describe the internal state of the medium, e.g., the content of various chemical substances in the medium, and the right sides φ_γ model the rates of change of the parameters q_γ, e.g., reaction rates. The quantity L_T is the entropy per unit volume. By the law of increase of entropy, the right sides φ_γ should satisfy the inequality $q_\gamma \varphi_\gamma \geqslant 0$ (we assume that $T > 0$).

Assuming that the unknown functions q_0, u_k, q_γ, and T are sufficiently smooth in the coordinates and time, from Eqs. (1.1) we can derive, as a consequence, one more equation, which is compatible with them. For this, we multiply equality (1.1a) by q_0, equalities (1.1b) by u_i, and equalities (1.1c) and (1.1d) by q_γ and T, respectively, and use the identities

$$q_0\, dL_{q_0} + u_i\, dL_{u_i} + q_\gamma\, dL_{q_\gamma} + T\, dL_T = dE, \tag{1.2}$$

$$q_0\, d(u_k L_{q_0}) + u_i\, d(u_k L)_{u_i} + q_\gamma\, d(u_k L)_{q_\gamma} + T\, d(u_k L)_T = d[u_k(E+L)],$$

where $E = q_0 L_{q_0} + u_i L_{u_i} + q_\gamma L_{q_\gamma} + T L_T - L$ is the Legendre transform of the potential L.

Using identities (1.2), we can transform the linear combination of Eqs. (1.1) with the chosen coefficients to the equality of divergent form

$$\frac{\partial E}{\partial t} + \frac{\partial [u_k(E+L)]}{\partial x_k} = 0. \tag{1.3}$$

In applied problems, this equality describes the law of conservation of energy. We note that here the zero right side is obtained by a coordinated choice of the right sides $-\varphi_\gamma$ and $q_\gamma \varphi_\gamma / T$ in Eqs. (1.1c) and (1.1d).

If the thermodynamic potential L is a convex function of its arguments, then (1.1) is a symmetric Friedrichs-hyperbolic system of equations, and this ensures correct (local) solvability of the Cauchy problem for smooth initial data. Indeed, denoting by r_i the unknowns $q_0, u_1, u_2, u_3, q_1, q_2, \ldots$, and T and by $M^{(k)}$ the products $M^{(k)} = u_k L$, we can write system (1.1)

$$\frac{\partial L_{r_i}}{\partial t} + \frac{\partial M^{(k)}_{r_i}}{\partial x_k} = -\psi_i$$

in equivalent form

$$L_{r_i r_j} \frac{\partial r_j}{\partial t} + M^{(k)}_{r_i r_j} \frac{\partial r_j}{\partial x_k} = -\psi_i \tag{1.4}$$

with symmetric matrices of coefficients composed of the derivatives $L_{r_i r_j}$ and $M^{(k)}_{r_i r_j}$. The convexity of L is equivalent to positive definiteness of the matrix of the coefficients at the derivatives with respect to t. By Friedrichs' definition, systems (1.4) are called hyperbolic.

For convenience, the law of conservation of momentum (1.1b) is written as

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L_{u_i} + \delta_{ik} L)}{\partial x_k} = 0. \tag{1.5}$$

We describe the transformation of Eqs. (1.1) and (1.3) in conversion to a moving coordinate system that moves at constant velocity relative to the initial system. Let the new coordinates y_k be related to the old coordinates by the equalities $y_k = x_k - U_k t \quad (U_k = \text{const})$, so that the new velocity components are $v_k = u_k - U_k$. The remaining unknowns $q_0, q_1, q_2, \ldots$, and T do not change.

Let

$$\tilde{L}(q_0, v_1, v_2, v_3, q_1, q_2, \ldots, T) = L(q_0, v_1 + U_1, v_2 + U_2, v_3 + U_3, q_1, q_2, \ldots, T).$$

In conversion to the moving coordinate system, equations of the form

$$\frac{\partial F}{\partial t} + \frac{\partial G_k}{\partial x_k} = -f$$

become

$$\frac{\partial F}{\partial t} + \frac{\partial (G_k - U_k F)}{\partial y_k} = -f.$$

Moreover, it is evident that

$$\tilde{L}_{q_0} = L_{q_0}, \qquad \tilde{L}_{v_k} = L_{u_k}, \qquad \tilde{L}_{q_\gamma} = L_{q_\gamma}, \qquad \tilde{L}_T = L_T.$$

Therefore, equalities (1.1a), (1.1c), and (1.1d) are written as

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial [(u_k L)_{q_0} - U_k L_{q_0}]}{\partial y_k} = 0,$$

$$\frac{\partial L_{q_\gamma}}{\partial t} + \frac{\partial [(u_k L)_{q_\gamma} - U_k L_{q_\gamma}]}{\partial y_k} = -\varphi_\gamma, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial [(u_k L)_T - U_k L_T]}{\partial y_k} = \frac{\varphi_\gamma q_\gamma}{T}$$

and after the substitution $u_k - U_k = v_k$, they take the form that differs from the initial form only in notation:

$$\frac{\partial \tilde{L}_{q_0}}{\partial t} + \frac{\partial (v_k \tilde{L})_{q_0}}{\partial y_k} = 0, \qquad \frac{\partial \tilde{L}_{q_\gamma}}{\partial t} + \frac{\partial (v_k \tilde{L})_{q_\gamma}}{\partial y_k} = -\varphi_\gamma, \qquad \frac{\partial \tilde{L}_T}{\partial t} + \frac{\partial (v_k \tilde{L})_T}{\partial y_k} = \frac{\varphi_\gamma q_\gamma}{T}.$$

Equations (1.1b), which describe the law of conservation of momentum, transform similarly. Equality (1.5) is written as

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial [(u_k L_{u_i} + \delta_{ik} L) - U_k L_{u_i}]}{\partial y_k} = 0,$$

and the law itself is finally written as

$$\frac{\partial \tilde{L}_{v_i}}{\partial t} + \frac{\partial (v_k \tilde{L}_{v_i} + \delta_{ik} \tilde{L})}{\partial y_k} = 0.$$

We now consider one more transformation that does not change the form of the equations entering into the "simplest" system (1.1). Neither the coordinate system nor the unknown functions, except for q_0, change in this transformation. The function q_0 is replaced by

$$Q_0 = q_0 - u_1 U_1 - u_2 U_2 - u_3 U_3 - K \qquad (K = \text{const}). \tag{1.6}$$

In this case,

$$L(q_0, u_1, u_2, u_3, q_1, q_2, \ldots, T)$$

$$= L(Q_0 + u_1 U_1 + u_2 U_2 + u_3 U_3 + K, u_1, u_2, u_3, q_1, q_2, \ldots, T) = \tilde{L}(Q_0, u_1, u_2, u_3, q_1, q_2, \ldots, T),$$

$$L_{q_0} = \tilde{L}_{Q_0}, \quad L_{u_k} = \tilde{L}_{u_k} - U_k \tilde{L}_{Q_0}, \quad L_{q_\gamma} = \tilde{L}_{q_\gamma}, \quad L_T = \tilde{L}_T.$$

Making the above change, we transform the law of conservation of mass (1.1a) into the equation

$$\frac{\partial \tilde{L}_{Q_0}}{\partial t} + \frac{\partial (u_k \tilde{L})_{Q_0}}{\partial x_k} = 0. \tag{1.7}$$

The momentum equations (1.5) are rearranged into the equations

$$\frac{\partial (\tilde{L}_{u_i} - U_i \tilde{L}_{Q_0})}{\partial t} + \frac{\partial [u_k (L_{u_i} - U_i \tilde{L}_{Q_0}) + \delta_{ik} \tilde{L}]}{\partial x_k} = 0,$$

which can be simplified if we write them in the form

$$\frac{\partial \tilde{L}_{u_i}}{\partial t} + \frac{\partial (u_k \tilde{L} u_i + \delta_{ik} \tilde{L})}{\partial x_k} - U_i \Big[\frac{\partial \tilde{L}_{Q_0}}{\partial t} + \frac{\partial (u_k \tilde{L})_{\tilde{Q}_0}}{\partial x_k} \Big] = 0$$

and discard the last terms equal to zero [see (1.7)].

Equations (1.1c) and (1.1d) are not changed by replacement of q_0 by Q_0 and L by $\tilde{L}$ and are written as

$$\frac{\partial \tilde{L}_{q_\gamma}}{\partial t} + \frac{\partial (u_k \tilde{L})_{q_\gamma}}{\partial x_k} = -\varphi_\gamma, \qquad \frac{\partial \tilde{L}_T}{\partial t} + \frac{\partial (u_k \tilde{L})_T}{\partial x_k} = \frac{\varphi_\gamma q_\gamma}{T}.$$

Thus, we have shown the invariance of system (1.1) under conversion to a coordinate system moving at constant velocity relative to the initial coordinate system. One can also assume that in conversion, the unknown function q_0, which enters into the mass conservation equation, also changes by the rule (1.6).

When converting to moving coordinates and keeping the standard designations u_i and q_0 for the new variables $u_i - U_i$ and $q_0 + u_i U_i - U_i U_i/2$, it is necessary, as was already noted, to change the formula for the generating potential L by taking into account its dependence on the conversion parameters U_i. The expression specifying the generating potential will not change if $L(q_0, u_1, u_2, u_3, q_1, q_2, \ldots, T) = \Lambda(q_0 + u_i u_i/2, q_1, q_2, \ldots, T)$.

Below we consider in detail the constraints that should be imposed on system (1.1) in order that it be invariant under orthogonal transformations (rotations) of the coordinate system, i.e., the constraints under which it is Galilean-invariant.

We consider one of the simplest examples of the equations of mathematical physics that admit representation in the form of (1.1).

The one-dimensional equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \qquad \frac{\partial(\rho u)}{\partial t} + \frac{\partial(p + \rho u^2)}{\partial x} = 0, \qquad \frac{\partial(\rho S)}{\partial t} + \frac{\partial(\rho S u)}{\partial x} = 0 \tag{1.8}$$

describe the motion of a gas with the energy equation of state $\mathcal{E} = \mathcal{E}(\rho, S)$, for which the pressure p and temperature T are determined by the formulas

$$p = \rho^2 \mathcal{E}_\rho(\rho, S), \qquad T = \mathcal{E}_S(\rho, S).$$

Equations (1.8) imply the law of conservation of energy

$$\frac{\partial(\rho[\mathcal{E}(\rho, S) + u^2/2])}{\partial t} + \frac{\partial(\rho u[\mathcal{E}(\rho, S) + p/\rho + u^2/2])}{\partial x} = 0. \tag{1.9}$$

To derive it, we need to multiply each of Eqs. (1.8) by the corresponding "integrating factor" $q_0 = \mathcal{E} + p/\rho - TS - u^2/2$, u, and T and sum up the products obtained. To make use of the necessary formalization, we introduce the notation $E = \rho[\mathcal{E}(\rho, S) + u^2/2]$ and define L so that

$$L_{q_0} = \rho, \qquad L_u = \rho u, \qquad L_T = \rho S,$$
$$E = q_0 L_{q_0} + u L_u + T L_T - L = (\mathcal{E} + p/\rho - TS - u^2/2)\rho + u\rho u + T\rho S - L$$
$$\equiv \rho\mathcal{E} + p + \rho u^2/2 - L = E + p - L.$$

Obviously, we should set $L = p = \rho^2 \mathcal{E}_\rho(\rho, S)$. In this case,

$$(uL)_{q_0} = u\rho, \quad (uL)_u = uL_u + L = \rho u^2 + p, \quad (uL)_T = uL_T = u\rho S.$$

So, if we define q_0, u, T, and L by means of the parametrization $q_0 = \mathcal{E}(\rho, S) + \rho\mathcal{E}_\rho(\rho, S) - S\mathcal{E}_S(\rho, S) - u^2/2$, $u = u$, $T = \mathcal{E}_S(\rho, S)$, and $L = \rho^2 \mathcal{E}_\rho(\rho, S)$, then Eqs. (1.8) and the law of conservation of energy (1.9) are written in divergent form

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(uL)_{q_0}}{\partial x} = 0, \qquad \frac{\partial L_u}{\partial t} + \frac{\partial(uL)_u}{\partial x} = 0,$$
$$\frac{\partial L_T}{\partial t} + \frac{\partial(uL)_T}{\partial x} = 0, \qquad \frac{\partial E}{\partial t} + \frac{\partial[u(E + L)]}{\partial x} = 0 \tag{1.10}$$
$$(E = q_0 L_{q_0} + u L_u + T L_T - L).$$

In the example considered, when we convert to the moving spatial coordinate $y = x - Ut$ with the velocity u replaced by v so that $u = v + U$, the required function

$$q_0 = \mathcal{E} + p/\rho - TS - u^2/2$$

should be replaced by

$$Q_0 = \mathcal{E} + p/\rho - TS - v^2/2 = q_0 + uU - U^2/2.$$

As a result, Eqs. (1.10) hold their form, but we should replace x by y, u by v, and q_0 by Q_0 in them.

The above example explains why in conversion to a moving coordinate system we needed to examine whether the attendant replacement of one of the required functions q_0 by $q_0 + u_k U_k + \text{const}$ is possible.

Systems of equations of the form of (1.1), which are basic for further constructions, describe processes in which viscous friction and diffusion are absent. If these should be taken into account, the equations should be modified.

We now present a modification of the one-dimensional version of system (1.1). The viscosity μ should be positive, and the matrix of the diffusion coefficients $D_{\gamma\beta}$ should be nonnegative definite (summation over repeated indices is performed):

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(uL)_{q_0}}{\partial x} = 0, \qquad \frac{\partial L_u}{\partial t} + \frac{\partial(L + uL_u - \mu\, \partial u/\partial x)}{\partial x} = 0,$$
$$\frac{\partial L_{q_\gamma}}{\partial t} + \frac{\partial(uL)_{q_\gamma}}{\partial x} = \frac{\partial}{\partial x}\Big(D_{\gamma\beta}\, \frac{\partial q_\gamma}{\partial x}\Big) - \varphi_\gamma, \tag{1.11}$$
$$\frac{\partial L_T}{\partial t} + \frac{\partial(uL)_T}{\partial x} = \frac{1}{T}\Big(\mu\, \frac{\partial u^2}{\partial x} + D_{\gamma\beta}\, \frac{\partial q_\gamma}{\partial x}\, \frac{\partial q_\beta}{\partial x} + q_\gamma \varphi_\gamma\Big).$$

The first and second equations in (1.11) describe the laws of conservation of mass and momentum, respectively, and the last equation describes the law of increase of entropy. For the solutions of system (1.11), the law of conservation of energy holds:

$$\frac{\partial E}{\partial t} + \frac{\partial[u(E+L) - \mu u\,\partial u/\partial x - q_\gamma D_{\gamma\beta}\,\partial q_\beta/\partial x]}{\partial x} = 0,$$
$$E = q_0 L_{q_0} + uL_u + q_\gamma L_{q_\gamma} + TL_T - L. \tag{1.12}$$

It is easy to check that as for Eqs. (1.1) and (1.3) considered above, the form of Eqs. (1.11) and (1.12) is retained in conversion to a new coordinate system moving at constant velocity relative to the initial coordinate system.

We give another example [more general than (1.11) and (1.12)] of one-dimensional conservation laws for multitemperature hydrodynamics confining ourselves to the variant with two temperatures. We chose this example after familiarizing ourselves with studies of [24–26].

Let the internal energy be the sum of the partial energies $\mathcal{E}^{(j)}(\rho, S_j)$, where S_j is the partial entropy ($j = 1, 2$). In this case we should set

$$L = \sum_j \rho^2 \mathcal{E}^{(j)}_S(\rho, S_j),$$

and take

$$q_0 = \sum_j [\mathcal{E}^{(j)}(\rho, S_j) + \rho\mathcal{E}^{(j)}_\rho(\rho, S_j)] - S_j\mathcal{E}_{S_j}(\rho, S_j) - \frac{u^2}{2}, \qquad T_j = \mathcal{E}^{(j)}_{S_j}$$

as the required functions. We have

$$L_{q_0} = \rho, \qquad L_u = \rho u, \qquad L_{T_j} = \rho S_j,$$

$$q_0 L_{q_0} + uL_u + \sum_j T_j L_{T_j} - L = \rho\left(\sum_j \mathcal{E}^{(j)}(\rho, S_j) + \frac{u^2}{2}\right) \equiv E.$$

By analogy with (1.11) and (1.12), the equations for one-dimensional viscous heat-conducting hydrodynamics can be written as

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(uL_{q_0})}{\partial x} = 0, \qquad \frac{\partial L_u}{\partial t} + \frac{\partial(L + uL_u - \mu\,\partial u/\partial x)}{\partial x} = 0,$$

$$\frac{\partial L_{T_1}}{\partial t} + \frac{\partial(uL_{T_1} - (K_1/T_1)\,\partial T_1/\partial x)}{\partial x} = \frac{K_1}{T_1^2}\Big(\frac{\partial T_1}{\partial x}\Big)^2 + a\,\frac{T_2 - T_1}{T_1} + \frac{\mu}{T_1}\Big(\frac{\partial u}{\partial x}\Big)^2,$$

$$\frac{\partial L_{T_2}}{\partial t} + \frac{\partial(uL_{T_2} - (K_2/T_2)\,\partial T_2/\partial x)}{\partial x} = \frac{K_2}{T_2^2}\Big(\frac{\partial T_2}{\partial x}\Big)^2 + a\,\frac{T_1 - T_2}{T_2},$$

$$\frac{\partial E}{\partial t} + \frac{\partial[u(E+L) - K_1\,\partial T_1/\partial x - K_2\,\partial T_2/\partial x - \mu u\,\partial u/\partial x]}{\partial x} = 0.$$

Here the last equality (law of conservation of energy) is obtained as a linear combination of all preceding equations taken with the coefficients q_0, u, T_1, and T_2, respectively. Additional terms $(a/T_1)(T_2 - T_1)$ and $(a/T_2)(T_1 - T_2)$ with positive coefficient a are included in the right sides of the third and fourth equations (entropy equations). These terms model the equalization of the temperatures T_1 and T_2. The multipliers $1/T_1$ and $1/T_2$ in these terms are chosen so that inclusion of these terms does not lead to violation of the law of conservation of energy. It is essential that summation of the entropy equations yields the equality

$$\frac{\partial[\rho(S_1 + S_2)]}{\partial t} + \frac{\partial[\rho u(S_1 + S_2) - (K_1/T_1)\,\partial T_1/\partial x - (K_2/T_2)\,\partial T_2/\partial x]}{\partial x}$$
$$= \frac{K_1}{T_1^2}\Big(\frac{\partial T_1}{\partial x}\Big)^2 + \frac{K_2}{T_2^2}\Big(\frac{\partial T_2}{\partial x}\Big)^2 + \frac{a}{T_1 T_2}(T_2 - T_1)^2 + \frac{\mu}{T_1}\Big(\frac{\partial u}{\partial x}\Big)^2,$$

which can be interpreted as the law of increase of total entropy. This increase is a consequence of the presence of the viscous stresses $\mu\,\partial u/\partial x$, the gradients of the temperatures T_1 and T_2, and the relaxation process, which yields a positive contribution if $T_2 \neq T_1$. Obviously, the construction described here can be automatically extended to a larger number of temperatures T_j.

Concluding the analysis of the simplest examples of account of dissipative processes that do not contradict the conservation laws for mass, momentum and energy and the law of increase of entropy, we note that the classification of the dissipative terms admitted by the Galilean invariance of equations is an important problem, which requires further investigation.

2. Invariance under Rotations and Galilean Transformations. We establish constraints on the "simplest" systems of equations (see Sec. 1) under which they describe processes independent of various rotations of the coordinate systems. Each such rotation is specified by an orthogonal real matrix $\mathcal{P}$ $(\mathcal{P}^{\mathrm{t}}\mathcal{P} = I_3)$ of order 3×3 with positive determinant $\det \mathcal{P} = +1$. The collection of these matrices constitutes the group $SO(3)$.

The transformation $\mathcal{P} \in SO(3)$ converts the coordinates x_1, x_2, and x_3 of a point x into the new coordinates y_1, y_2, and y_3 by the formula

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \mathcal{P} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathcal{P}\boldsymbol{x}.$$

The old coordinates can be calculated in terms of the new coordinates using the inverse matrix $\mathcal{P}^{-1} = \mathcal{P}^{\mathrm{t}}$:

$$\boldsymbol{x} = \mathcal{P}^{-1}\boldsymbol{y} = \mathcal{P}^{\mathrm{t}}\boldsymbol{y}.$$

After the transformation $\mathcal{P}$, the velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} u_1(x_1, x_2, x_3, t) \\ u_2(x_1, x_2, x_3, t) \\ u_3(x_1, x_2, x_3, t) \end{pmatrix}$$

is given by the vector-function $\boldsymbol{v}(\boldsymbol{y}, t)$ calculated in terms of $\boldsymbol{u}(\boldsymbol{x}, t)$ by the formula

$$\boldsymbol{v}(\boldsymbol{y}, t) = \mathcal{P}\boldsymbol{u}(\mathcal{P}^{-1}\boldsymbol{y}, t).$$

We now unite the unknown functions that describe the state of the medium into a vector-function $\boldsymbol{q} = (q_1, q_2, \ldots)^{\mathrm{t}}$, which should also be transformed by rotations of the coordinate system. Each coordinate transformation described by the matrix $\mathcal{P}$ should correspond to an orthogonal matrix $\Omega = \Omega(\mathcal{P})$, which converts a vector $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x}, t)$ into $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{y}, t) = \Omega\boldsymbol{q}(\mathcal{P}^{-1}\boldsymbol{y}, t)$. In successive transformations $\boldsymbol{y} = \mathcal{P}_1\boldsymbol{x}$ and $\boldsymbol{z} = \mathcal{P}_2\boldsymbol{y} = \mathcal{P}_2\mathcal{P}_1\boldsymbol{x}$, the unknown vectors are transformed successively $(\boldsymbol{q} \to \boldsymbol{p} \to \boldsymbol{r})$ using the corresponding orthogonal matrices Ω_1 and Ω_2:

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{z}, t) = \Omega_2\boldsymbol{p}(\mathcal{P}_2^{-1}\boldsymbol{z}, t),$$

$$\boldsymbol{p}(\mathcal{P}_2^{-1}\boldsymbol{z}, t) = \boldsymbol{p}(\boldsymbol{y}, t) = \Omega_1\boldsymbol{q}(\mathcal{P}_1^{-1}\boldsymbol{y}, t) = \Omega_1\boldsymbol{q}(\mathcal{P}_1^{-1}\mathcal{P}_2^{-1}\boldsymbol{z}, t),$$

$$\boldsymbol{r} = \Omega_2\Omega_1\boldsymbol{q}(\mathcal{P}_1^{-1}\mathcal{P}_2^{-1}\boldsymbol{z}, t).$$

The correspondence between transformations $\mathcal{P}$ of the spatial coordinates and the corresponding transformations Ω of the unknown vector-function $\boldsymbol{q}(\boldsymbol{x}, t)$ should be a representation of the group $SO(3)$ by orthogonal matrices.

The generating potential $L = L(\boldsymbol{u}, \boldsymbol{q}, T)$ will be assumed to be invariant, so that

$$L(\boldsymbol{u}, \boldsymbol{q}, T) = L(\mathcal{P}\boldsymbol{u}, \Omega\boldsymbol{q}, T) = L(\boldsymbol{v}, \boldsymbol{p}, T). \tag{2.1}$$

When investigating one or another concrete system, we shall assume that the finite-dimensional orthogonal representation $\Omega(\mathcal{P})$ which keeps the generating potential invariant for every $\mathcal{P} \in SO(3)$ is known.

In quantum-mechanical problems of mathematical physics, not only orthogonal but also unitary representations can be used. In this case, the unknowns q_j should be assumed to be complex. In Sec. 3 we shall consider possible expedients that allow us to use the analysis of invariance of equations in the case of such generalizations as well.

Whatever the function $f(\boldsymbol{y}, t)$ may be:

$$f(\boldsymbol{y}, t) = f(y_1, y_2, y_3, t) = f(\mathcal{P}\boldsymbol{x}, t) = f(\mathcal{P}_{1j}x_j, \mathcal{P}_{2j}x_j, \mathcal{P}_{3j}x_j, t),$$

the following equalities are valid by virtue of the orthogonality of the transformation $\mathcal{P}$:

$$f_{x_j} = f_{y_i}\mathcal{P}_{ij}, \quad v_i = \mathcal{P}_{ij}u_j, \quad u_j = \mathcal{P}_{ji}v_i, \quad u_i f_{x_i} \equiv u_j f_{x_j} = v_i f_{y_i}\mathcal{P}_{ji}\mathcal{P}_{ij} = v_i f_{y_i}. \tag{2.2}$$

Moreover, scalars, in particular, the temperature T, do not change in this transformation.

The equality (for $dT = 0$)

$$dL = L_{u_i}\,du_i + L_{q_\gamma}\,dq_\gamma = L_{v_j}\mathcal{P}_{ij}\,du_i + L_{p\beta}\Omega_{\beta\alpha}\,dq_\alpha$$

and the orthogonality of $\mathcal{P}$ and Ω imply that

$$\begin{gathered} L_{u_i} = \mathcal{P}_{ji}L_{v_j}, \quad L_u = \mathcal{P}^{\mathrm{t}}L_v, \quad L_v = \mathcal{P}L_u, \\ L_{q_\alpha} = \Omega_{\beta\alpha}L_{p_\beta}, \quad L_q = \Omega^{\mathrm{t}}L_p, \quad L_p = \Omega L_q. \end{gathered} \tag{2.3}$$

For every scalar function $g(\boldsymbol{y},t) = g(y_1,y_2,y_3,t) = g(\mathcal{P}_{1j}x_j,\mathcal{P}_{2j}x_j,\mathcal{P}_{3j}x_j,t)$, its gradients in the coordinate systems $\boldsymbol{x}$ and $\boldsymbol{y}$ are linked by the equality

$$\frac{\partial g}{\partial \boldsymbol{y}} = \begin{pmatrix} g_{y_1} \\ g_{y_2} \\ g_{y_3} \end{pmatrix} = \mathcal{P}\begin{pmatrix} g_{x_1} \\ g_{x_2} \\ g_{x_3} \end{pmatrix} = \mathcal{P}\,\frac{\partial g}{\partial \boldsymbol{x}}. \tag{2.4}$$

We consider the system of equations specified by the invariant [see (2.1)] potential L:

$$\begin{gathered} \frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0, \quad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial(u_k L_{u_i})}{\partial x_k} + \frac{\partial L}{\partial x_i} = 0, \\ \frac{\partial L_{q_\gamma}}{\partial t} + \frac{\partial(u_k L_{q_\gamma})}{\partial x_k} = -\varphi_\gamma, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{q_\gamma\varphi_\gamma}{T}. \end{gathered} \tag{2.5}$$

We assume that in rotations of the coordinate system, the vector $\boldsymbol{q}$ with components q_γ is transformed by a certain representation of the rotation group. The scalar quantities q_0 and T, which do not change in rotations, can be included in the number of the components of the vector $\boldsymbol{q}$, assuming that $\varphi_0 = 0$ and $\varphi_T = q_\gamma\varphi_\gamma/T$.

It follows from (2.2)–(2.4) that the orthogonal transformations $\mathcal{P}$ and Ω

$$\boldsymbol{y} = \mathcal{P}\boldsymbol{x}, \quad \boldsymbol{v}(\boldsymbol{y},t) = \mathcal{P}\boldsymbol{u}(\mathcal{P}^{-1}\boldsymbol{y},t), \quad \boldsymbol{p}(\boldsymbol{y},t) = \Omega\boldsymbol{q}(\mathcal{P}^{-1}\boldsymbol{y},t),$$

$$\boldsymbol{p}_0(\boldsymbol{y},t) = \boldsymbol{q}_0(\mathcal{P}^{-1}\boldsymbol{y},t), \quad \hat{T} = T(\mathcal{P}^{-1}\boldsymbol{y},t), \quad \hat{\varphi}_\gamma(\boldsymbol{y},t) = \varphi_\gamma(\mathcal{P}^{-1}\boldsymbol{y},t)$$

convert system (2.5) into a system of equations that differs from (2.5) only in notation:

$$\frac{\partial L_{p_0}}{\partial t} + \frac{\partial(v_k L_{p_0})}{\partial x_k} = 0, \quad \frac{\partial L_{v_i}}{\partial t} + \frac{\partial(v_k L_{v_i})}{\partial x_k} + \frac{\partial L}{\partial x_i} = 0,$$

$$\frac{\partial L_{p_\gamma}}{\partial t} + \frac{\partial(v_k L_{p_\gamma})}{\partial x_k} = -\hat{\varphi}_\gamma, \quad \frac{\partial L_{\hat{T}}}{\partial t} + \frac{\partial(v_k L_{\hat{T}})}{\partial x_k} = \frac{p_\gamma\hat{\varphi}_\gamma}{\hat{T}}.$$

This suggests that system (2.5) is invariant under rotations under the above assumptions of invariance of the generating thermodynamic potential L, which depends on the unknown functions — vectors that are rotationally transformed by irreducible representations.

System (2.5) coincides with the "simplest" system (1.1) introduced in Sec. 1 and can also be supplemented with the equality compatible with it:

$$\frac{\partial E}{\partial t} + \frac{\partial[u_k(E+L)]}{\partial x_k} = 0, \qquad E = q_0L_{q_0} + u_iL_{u_i} + q_\gamma L_{q_\gamma} + TL_T - L. \tag{2.6}$$

It was shown in Sec. 1 that the system of equations considered is invariant under conversion to a coordinate system moving at constant velocity if the generating potential is given in the form $L = \Lambda(q_0 + u_iu_i/2, q_1, q_2, \dots, T)$.

Thus, by virtue of the invariance of the equations under rotations of the coordinate systems (see Sec. 1), we can assert that system (1.1), (1.3) or system (2.5), (2.6), which coincides with it, is invariant under arbitrary Galilean transformations. Obviously, it is also invariant under a parallel translation of the coordinate system involving shift of its origin.

3. Concrete Definition of the Required Functions in the Systems under Investigation. In Secs. 1 and 2, we have described the structure of a Galilean-invariant special system of partial differential equations which contains conservation laws for mass, momentum, and energy and the law of conservation (or increase) of entropy. In this system, the unknown functions are the velocity field $\boldsymbol{u} \equiv (u_1, u_2, u_3)$ and the field of a certain, as a rule, multicomponent vector $\boldsymbol{q}$ that is transformed in rotations by an orthogonal representation of the rotation group or its universal covering group $SU(2)$ (multiplicative group of quaternions).

We change the numbering of the independent variables x_1, x_2, and x_3 and the velocity components u_1, u_2, and u_3. For further considerations, it is more convenient to use the notation x_{-1}, x_0, and x_1 and u_{-1}, u_0, and u_1, respectively.

We assume that the vector $\boldsymbol{q}$ is compound, and its parts are "vector components" — the vectors $\boldsymbol{q}^{(A_1)}$, $\boldsymbol{q}^{(A_2)}, \ldots$, each of which is transformed by an irreducible orthogonal representation of the corresponding weight A_j.

Among the vector components there may be more than one component with the same weight A.

If the weight A is integer, the vector $\boldsymbol{q}^{(A)}$ should have odd number $(2A+1)$ of components. For these components — real numbers $q_a^{(A)}$ — we use integer numbering: $a = -A$, $-A+1$, ..., -1, 0, 1, ..., $A-1$, and A. In this case, the zero weight $A = 0$ corresponds to one scalar component $q_0^{(0)}$, which does not change in rotations.

In the case of a half-integer weight A, it is common to use unitary representations in a complex vector space of even dimension $2A+1$. In this space, each vector $\boldsymbol{q}^{(A)}$ has $2A+1$ complex components $q_a^{(A)} + ir_a^{(A)}$ ($a = -A$, $-A+1$, ..., $-1/2$, $1/2$, ..., $A-1$, and A), and the representation itself induces in natural way an orthogonal representation in the space of $2(2A+1)$-dimensional real vectors $\boldsymbol{q}^{(A)}$ with the components q_{-A}, q_{-A+1}, ..., and q_A and r_{-A}, r_{-A+1}, ..., and r_A. In the examples below, we use exactly this realization of orthogonal representations of the group $SU(2)$ or the rotation group $SO(3)$. In the case of representations of half-integer weight, each rotation from $SO(3)$ through an angle of 2π about a certain axis corresponds to a transformation of the vector $\boldsymbol{q}^{(A)}$ into the vector $-\boldsymbol{q}^{(A)}$, whose all components $-q_a^{(A)}$ and $-r_a^{(A)}$ are opposite in signs to the components $q_a^{(A)}$ and $r_a^{(A)}$ of the initial vector $\boldsymbol{q}^{(A)}$. For example, the vector variable $\boldsymbol{q}^{(1/2)}$, which is transformed by an orthogonal (two-valued) representation of weight 1/2, has four components $q_{-1/2}$, $r_{-1/2}$, $q_{1/2}$, and $r_{1/2}$. Each rotation $g \in SO(3)$ given by the matrix $g = g_0(\psi_0)g_{-1}(\theta_{-1})g_0(\varphi_0)$ represented as the product of rotations around the coordinate axes indexed "zero," "minus one," and again "zero" corresponds to the orthogonal transformation of the vector $\boldsymbol{q}^{(1/2)}$ described by the matrix

$$\Omega^{1/2}(\psi_0, \theta_{-1}, \varphi_0) = \begin{pmatrix} \cos\frac{\theta_{-1}}{2}\cos\frac{\varphi_0+\psi_0}{2} & \sin\frac{\theta_{-1}}{2}\cos\frac{\varphi_0-\psi_0}{2} & \cos\frac{\theta_{-1}}{2}\sin\frac{\varphi_0+\psi_0}{2} & -\sin\frac{\theta_{-1}}{2}\sin\frac{\varphi_0-\psi_0}{2} \\ -\sin\frac{\theta_{-1}}{2}\cos\frac{\varphi_0-\psi_0}{2} & \cos\frac{\theta_{-1}}{2}\cos\frac{\varphi_0+\psi_0}{2} & -\sin\frac{\theta_{-1}}{2}\sin\frac{\varphi_0-\psi_0}{2} & -\cos\frac{\theta_{-1}}{2}\sin\frac{\varphi_0+\psi_0}{2} \\ -\cos\frac{\theta_{-1}}{2}\sin\frac{\varphi_0+\psi_0}{2} & \sin\frac{\theta_{-1}}{2}\sin\frac{\varphi_0-\psi_0}{2} & \cos\frac{\theta_{-1}}{2}\cos\frac{\varphi_0+\psi_0}{2} & \sin\frac{\theta_{-1}}{2}\cos\frac{\varphi_0-\psi_0}{2} \\ \sin\frac{\theta_{-1}}{2}\sin\frac{\varphi_0-\psi_0}{2} & \cos\frac{\theta_{-1}}{2}\sin\frac{\varphi_0+\psi_0}{2} & -\sin\frac{\theta_{-1}}{2}\cos\frac{\varphi_0-\psi_0}{2} & \cos\frac{\theta_{-1}}{2}\cos\frac{\varphi_0+\psi_0}{2} \end{pmatrix}.$$

Each rotation $g \in SO(3)$, $g = g_0(\psi_0)g_{-1}(\theta_{-1})g_0(\varphi_0)$ is a rotation through angle $\omega = \psi_0$ around the axis into which the x_0 axis is converted by successive rotations $g_0(\varphi_0)$ and $g_{-1}(\theta_{-1})$. It should be noted that rotation through an angle $\psi_0 + 2\pi$ around the same axis, i.e., rotation with the set of parameters $\psi_0 + 2\pi$, θ_{-1}, and φ_0, corresponds to $\Omega^{1/2}(\psi_0 + 2\pi, \theta_{-1}, \varphi_0) = -\Omega^{1/2}(\psi_0, \theta_{-1}, \varphi_0)$.

The above example of an orthogonal representation is obtained from a 2D unitary representation, in which the 2D complex vector

$$\boldsymbol{q}^{(1/2)} = \begin{bmatrix} q_{-1/2} + ir_{-1/2} \\ q_{1/2} + ir_{1/2} \end{bmatrix}$$

is transformed using unitary matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \alpha = \mathrm{e}^{-i(\varphi_0+\psi_0)/2}\cos\frac{\theta_{-1}}{2}, \quad \beta = \mathrm{e}^{i(\varphi_0-\psi_0)/2}\sin\frac{\theta_{-1}}{2}.$$

As an example of a Galilean-invariant system in which part of the unknowns, namely, T, q_0, u_{-1}, u_0, and u_1 are transformed by rotations according to one-valued representations (of weights 0 and 1), and the other part is transformed by a two-valued representation (of weight 1/2), we consider the equations generated by a certain potential

$$L = L(q_0; u_{-1}, u_0, u_1; q_{-1/2}, r_{-1/2}, q_{1/2}, r_{1/2}, T) = \Lambda(q_0 + u_i u_i/2, q_{-1/2}, r_{-1/2}, q_{1/2}, r_{1/2}, T),$$

which is invariant under rotations:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} = 0,$$

$$\frac{\partial L_{q_{-1/2}}}{\partial t} + \frac{\partial (u_k L_{q_{-1/2}})}{\partial x_k} = -r_{1/2}, \qquad \frac{\partial L_{q_{1/2}}}{\partial t} + \frac{\partial (u_k L_{q_{1/2}})}{\partial x_k} = r_{-1/2},$$

$$\frac{\partial L_{r_{-1/2}}}{\partial t} + \frac{\partial (u_k L_{r_{-1/2}})}{\partial x_k} = -q_{1/2}, \qquad \frac{\partial L_{r_{1/2}}}{\partial t} + \frac{\partial (u_k L_{r_{1/2}})}{\partial x_k} = q_{-1/2},$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial (u_k L_T)}{\partial x_k} = 0, \qquad \frac{\partial E}{\partial t} + \frac{\partial [u_k (E + L)]}{\partial x_k} = 0,$$

$$E = u_k L_{u_k} + q_j L_{q_j} + r_j L_{r_j} + T L_T - L.$$

The results obtained in Secs. 1 and 2 show that the system considered is compatible, although overdetermined since it has 10 equations for 9 unknowns functions. The zero right side in the entropy equation was obtained by a special choice of the right sides.

The short notation $\boldsymbol{q}^{(A)}$ is used for a vector with components $q_a^{(A)}$ that is transformed by an irreducible representation of weight A. Sometimes, it is more convenient to deal with vector-functions represented by divalent tensors $q_{k\alpha}^{(1,A)}$ with the second subscript a Greek letter and the first subscript a Roman letter. Each rotation $\mathcal{P}^{(1)}$ of the coordinate system in the representation of weight A corresponds to an orthogonal transformation (representation) $\Omega^{(A)}$ given by a real matrix which has order $(2A+1) \times (2A+1)$ for integer A and order $[2(2A+1)] \times [2(2A+1)]$ for half-integer A. Obviously, the Greek subscript α runs through the corresponding number of various values, whereas the Roman subscript k takes only three values (for the chosen numbering, $k = -1$, 0, and 1).

The tensor $q_{k\alpha}^{(1,A)}$ is transformed by the rule $[q_{k\alpha}^{(1,A)}]' = \Omega_{\alpha\beta}^{(A)} \mathcal{P}_{kj}^{(1)} q_{j\beta}^{(1,A)}$. The representation generated by such transformations is reducible if $A \neq 0$ and, as is known, it is decomposed into irreducible representations.

As an example, we give a compatible Galilean-invariant overdetermined system, where q_0 is a scalar, u_j are the velocity components, $r_{j\alpha}$ are the components of a tensor $r^{(1,1)}$, and T is the temperature:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} = 0,$$

$$\frac{\partial L_{r_{i\alpha}}}{\partial t} + \frac{\partial (u_k L_{r_{k\alpha}})}{\partial x_k} = -\varphi_{i\alpha}, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial (u_k L_T)}{\partial x_k} = \frac{r_{j\alpha}\varphi_{j\alpha}}{T},$$

$$\frac{\partial E}{\partial t} + \frac{\partial [u_k (E + L)]}{\partial x_k} = 0, \qquad E = q_0 L_{q_0} + u_i L_{u_i} + r_{j\alpha} L_{r_{j\alpha}} + T L_T - L.$$

The potential L, which generates this system, is an invariant function of all the unknowns q_0, u_i, $r_{j\alpha}$, and T. The right sides $-\varphi_{i\alpha}$ should be determined so as not to violate the invariance. The equations with zero right sides can be regarded as exact conservation laws for mass, momentum, and energy.

In conclusion we give matrices $\Omega^{(N)}(\psi_0, \theta_{-1}, \varphi_0) = \Omega^{(N)}(0, 0, \psi_0)\Omega^{(N)}(0, \theta_{-1}, 0)\Omega^{(N)}(0, 0, \varphi_0)$, which in the standard canonical basis give the representation (of integer weight N) of the rotations $\mathcal{P}(\psi_0, \theta_{-1}, \varphi_0)$ specified by Euler's angles φ_0, θ_{-1}, and ψ_0. The subscripts correspond to the numbers of the coordinate axes around which the rotation is performed. It should be noted that $\mathcal{P}(\psi_0, \theta_{-1}, \varphi_0) = \Omega^{(1)}(\psi_0, \theta_{-1}, \varphi_0)$.

The matrix elements corresponding to the rotation φ_0 are calculated from the formulas

$$\Omega_{m,m}^{(N)}(\varphi_0, 0, 0) = \Omega_{m,m}^{(N)}(0, 0, \varphi_0) = \cos m\varphi_0, \quad \Omega_{-m,m}^{(N)}(\varphi_0, 0, 0) = \Omega_{-m,m}^{(N)}(0, 0, \varphi_0) = \sin m\varphi_0,$$

if $N \geqslant |m| \geqslant 1$ and $\Omega_{0,0}^{(N)}(0, 0, \varphi_0) = 1$ or $\Omega_{k,m}^{(N)}(0, 0, \varphi_0) = 0$ if $|k| \neq |m|$.

It is convenient to express the elements $\Omega^{(N)}_{km}(0,\theta_{-1},0) = (-1)^{k+m}\Omega^{(N)}_{mk}(0,\theta_{-1},0)$ in terms of $\mu = \cos\theta_{-1}$ ($1 \leqslant k$ and $m \leqslant N$):

$$\Omega^{(N)}_{\pm k,\pm m}(0,\theta_{-1},0) = \frac{(-1)^{N+m}}{2^N(1-\mu^2)^{(k+m)/2}}\sqrt{\frac{(N+m)!}{(N-k)!(N+k)!(N-m)!}}$$

$$\times\Big\{(1-\mu)^k\,\frac{d^{N-m}}{d\mu^{N-m}}\,[(1+\mu)^{N+k}(1-\mu)^{N-k}] \pm (-1)^k(1+\mu)^k\,\frac{d^{N-m}}{d\mu^{N-m}}\,[(1+\mu)^{N-k}(1-\mu)^{N+k}]\Big\},$$

$$\Omega^{(N)}_{k,0}(0,\theta_{-1},0) = (-1)^k\Omega^{(N)}_{0,k}(0,\theta_{-1},0) = \frac{(-1)^{N+1}}{N!2^N}\sqrt{\frac{2(N+k)!}{(N-k)!}}\,\frac{1}{(1-\mu^2)^{k/2}}\,\frac{d^{N-k}}{d\mu^{N-k}}\,(1-\mu^2)^N,$$

$$\Omega^{(N)}_{0,0}(0,\theta_{-1},0) = \frac{(-1)^N}{N!2^N}\,\frac{d^N}{d\mu^N}\,(1-\mu^2)^N,$$

$$\Omega^{(N)}_{-k,m}(0,\theta_{-1},0) = \Omega^{(N)}_{k,-m}(0,\theta_{-1},0) = \Omega^{(N)}_{-k,0}(0,\theta_{-1},0) = \Omega^{(N)}_{0,-m}(0,\theta_{-1},0) = 0.$$

Conclusions. This paper reports results of investigation of the hyperbolicity, Galilean invariance, and thermodynamical consistency of the simplest conservation laws encountered in mathematical physics.

A special choice of unknown vector-functions that are transformed by irreducible orthogonal representations of the rotation group is also proposed. The use of these vector-functions as variables will simplify the study (which was begun in [15, 16]) of the relationship between the theory of group representations and the systematization of the differential equations describing evolutionary processes in various continuous media.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 01-01-00766).

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